GRAVITATIONAL FIELD OF DISTANT ROTATING MASSES

RAMESH. V. WAGH.

APPLIED PHYSICS DEPARTMENT, GOVERNMENT POLYTECHNIC, POONA-5

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ABSTRACT. In 1958, Thirring calculated the gravitational field near the centre of a rotating spherical shell. The case can be generalised to that of a rotating mass, where the field away from the mass can be determined. Thirring assumed

$$T^{\mu\nu} = \rho v^{\mu} v^{\nu} \tag{i}$$

But it can be shown that, starting from a Galilean field we can build up the case a non-Galilean field by introducing some small term in the metric tensor whose Galilean value is unity. Then by a straight forward process, we calculate $T^{\mu\nu}$ given by

$$T^{\mu\nu} = (p_0+\rho_{00})v^{\mu}v^{\nu} - p_0 g^{\mu\nu} \tag{ii}$$

By the introduction of a rotating mass in the galilean field, which now slightly deviates from its original characteristics is given by the metric tensor

$$\left.\begin{aligned} g_{11},\ g_{22},\ g_{33} &= -(1+\alpha)(1,\ r^2,\ r^2 \sin^2\theta) \\ \text{and} \quad g_{44} &= (1-\alpha) \end{aligned}\right\} \tag{iii}$$

The energy-momentum tensor is then calculated with $\alpha = m(1-r^2 \sin^2\theta\omega^2)$ and it is shown that this satisfies the conditions of mechanics. It is incidentally shown that there is no necessity of introducing $E^{\mu\nu}$ in the expression for $T^{\mu\nu}$, as recently done by Bass and Pirani (1955). The desired results are obtained without making such assumptions.

1. INTRODUCTION AND DERIVATION OF ENERGY—MOMENTUM TESNOR

In a recent paper, Bass and Pirani (1955) and others (1956) have discussed corrections to Thirring's calculations of the gravitational field near the centre of of a rotating spherical shell by introducing an additional term $E^{\mu\nu}$ representing the elastic interaction between particles of the shell, in the expression for energy momentum tensor $T^{\mu\nu} = \rho v^{\mu} v^{\nu}$, where the symbols on the right hand side have their usual meanings. The method followed by them is an indirect one in so far as addition of $E^{\mu\nu}$ is made to derive the expression for the energy momentum tensor and then certain assumptions are made as to the behaviour of $E^{\mu\nu}$ with reference to the rotating shell. We can however derive the more general expression for the energy—momentum tensor in the form

$$T^{\mu\nu} = (p_0+\rho_{00})v^{\mu}v^{\nu} - g^{\mu\nu}p_0 \tag{1}$$

wherein we can still assume Galilean values for $g_{\mu\nu}$ or $g^{\mu\nu}$. With this and the modifications introduced in the field by the rotating mass, the whole problem can be worked out in a more straight forward manner as given below.

Accordingly, the flat space-time as envisaged in the general theory of relativity defined by the line-element,

$$ds^2 = -\{(dx_0^1)^2+(dx_0^2))^2+(dx_0^3)^2+(dx_0^4)^2\} \tag{2}$$

or its analogue in spherical polar co-ordinates namely,

$$ds^2 = -(dx'^1_0)-r^2(dx'^2_0)-r^2 \sin^2\theta(dx'^3_0)^2-(dx'^4_0)^2 \tag{2'}$$

We shall use this in preference to (2) and remove the dashes in the equations that follow. By suitable transformation of dx_0^4 the 'negative' sign for ρ_{00} which may otherwise appear can be avoided. Energy-momentum tensor can be expressed as :

$$T_0^{\alpha\beta} = \begin{vmatrix} -p_0 & 0 & 0 & 0 \\ 0 & -p_0 & 0 & 0 \\ 0 & 0 & -p_0 & 0 \\ 0 & 0 & 0 & \rho_{00} \end{vmatrix} \quad \ldots \quad (3)$$

where $p^0_{xx} = -p_0$, $p^0_{xy} = 0$ etc, etc. as given in equation 85.1, p. 215 of Tolman's Relativity Thermodynamics and Cosmology (1934). If by the appearance of the rotating spherical mass at great distances from the point of interest, the metric for the gravitational field is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad \ldots \quad (4)$$

then the energy-momentum tensor for this case can be written down in accordance with the usual transformation law as

$$T^{\mu\nu} = \frac{\partial x^\mu}{\partial x_0^\alpha} \cdot \frac{\partial x^\nu}{\partial x_0^\beta} \cdot T_0^{\alpha\beta} \quad \ldots \quad (5)$$

Considering Eqns. (2), (3) and (5) we have,

$$T^{\mu\nu} = -\frac{\partial x^\mu}{\partial x_0^1} \cdot \frac{\partial x^\nu}{\partial x_0^1} p_0 - \frac{\partial x^\mu}{\partial x_0^2} \cdot \frac{\partial x^\nu}{\partial x_0^2} p_0 - \frac{\partial x^\mu}{\partial x_0^3} \cdot \frac{\partial x^\nu}{\partial x_0^3} p_0 + \rho_{00} \frac{\partial x^\mu}{\partial x_0^4} \cdot \frac{\partial x^\nu}{\partial x_0^4} \quad \ldots \quad (6)$$

Here, the continuity of the $g_{\mu\nu}$ in the sense of three spaces in space-time has not been considered. (See, for instance a paper "Discontinuities in spherically symmetric gravitational fields and shells of radiation" by W. Israel where reference in the subject are given.)

Let R_0 be the radius of the rotating shell thus formed and let Ω be its angular velocity of rotation. It may also be assumed that due to the presence of the rotating masses the field deviates only slightly from that in a flat space-time. This will be satisfied approximately if α is considered to be a small quantity of first order.

$$v_0^1 = \frac{dx_0^1}{ds} = 0; \; v_0^2 = \frac{dx_0^2}{ds} = 0; \; \frac{v_0^3}{v_0^4} = \frac{\Omega}{i} \Bigg\} \qquad \ldots (7)$$

Initially, we have and $(v_0^4)^2 + R_0^2 \sin^2\theta (v_0^3)^2 = -1$

where $$v_0^3 = \frac{dx_0^3}{ds} \text{ and } v_0^4 = \frac{dx_0^4}{ds}$$

The third and the fourth of Eq. (7) above give

$$\left.\begin{aligned} & v_0^3 = \Omega\,(1 - R_0^2\Omega^2 \sin^2\theta)^{-\frac{1}{2}} \\ \text{and} \quad & v_0^4 = i(1 - R_0^2\Omega^2 \sin^2\theta)^{-\frac{1}{2}} \end{aligned}\right\} \qquad \ldots (8)$$

The metric tensor is now given by the transformation

$$g^{\mu\nu} = \frac{\partial x^\mu}{\partial x_0^\alpha} \cdot \frac{\partial x^\nu}{\partial x_0^\beta} \cdot g_0^{\alpha\beta}$$

which for $g_0^{\alpha\beta}$ given by the metric (2) reduces to

$$g^{\mu\nu} = -\frac{\partial x^\mu}{\partial x_0^1} \cdot \frac{\partial x^\nu}{\partial x_0^1} - \frac{\partial x^\mu}{\partial x_0^2} \cdot \frac{\partial x^\nu}{\partial x_0^2} - \frac{\partial x^\mu}{\partial x_0^3} \cdot \frac{\partial x^\nu}{\partial x_0^3} - \frac{\partial x^\mu}{\partial x_0^4} \cdot \frac{\partial x^\nu}{\partial x_0^4} \qquad \ldots (9)$$

We can also write the velocity components as

$$\frac{dx^\mu}{ds} = \frac{\partial x^\mu}{\partial x_0^\alpha} \cdot \frac{dx_0^\alpha}{ds} = \frac{\partial x^\mu}{\partial x_0^\alpha} \cdot v_0^\alpha \,.$$

substituting for v_0^α from Eq. (7) we get

$$\frac{dx^\mu}{ds} = 2i(1 - R_0^2\,\Omega^2 \sin^2\theta)^{-\frac{1}{2}} \frac{\partial x^\mu}{\partial x_0^4} \qquad \ldots (10)$$

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with the help of Eqns. (6) and (9) the energy-momentum tensorc an now be written as

$$T^{\mu\nu} = (p_0+\rho_{00}) \frac{\partial x^\mu}{\partial x_0{}^4} \cdot \frac{\partial x^\nu}{\partial x_0{}^4} + g^{\mu\nu} p_0 \qquad \dots \quad (11)$$

The positive sign for the second term on the right hand side is due to the particular choice of the metric tensor as indicated in Eq (2). Substituting for $\frac{\partial x^\mu}{\partial x_0{}^4}$ in Eq. (11), we get

$$T^{\mu\nu} = -\tfrac{1}{4}(p_0+\rho_{00})(1-R_0{}^2\,\Omega^2 \sin^2\theta) v^\mu v^\nu + g^{\mu\nu} p_0 \qquad \dots \quad (12)$$

It is usual to put $idt = dx_0{}^4$ and $t' = t$, where t' is the time for the modified metric We can, therefore, rewrite Eq. (10) as

$$\frac{dx^\mu}{ds} = 2(1-R_0{}^2\,\Omega^2 \sin^2\theta_0)^{-\frac{1}{2}} \frac{\partial x^\mu}{\partial t} \qquad \dots \quad (13)$$

Where ∂_0 is a certain given value of θ.

2. THE METRIC

It is assumed that the metric inside the rotating mass is only slightly modified and it does not differ very much from the Galilean field. We may, therefore, write

$$ds^2 = -(1+\alpha)(dz^2+r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + (1-\alpha) dt^2 \qquad \dots \quad (14)$$

where $\alpha = \alpha(r, \theta, t)$. α is an infinitesimal of the first order and its first and second order variations are also small. Their products are infinitesimals of higher order. In the following derivation these are neglected.

It would appear that the solution may be similar to that obtained from Schwarzschild's exterior line-element where it assumed that at large distances from the central particle the space would have symmetrical and isotropic properties. We now proceed to obtain the solution in this case.

For the metric given by (14) the energy-momentum tensor has the following surviving components to the first order of small quantities :

$$KT^1{}_1 = \frac{1}{r}\frac{\partial \alpha}{\partial r} + \frac{\partial^2 \alpha}{\partial t^2} \qquad \dots \quad (15)$$

$$KT_2{}^2 = KT_3{}^3 = -\frac{1}{r}\frac{\partial \alpha}{\partial r} + \frac{\partial^2 \alpha}{\partial t^2} \qquad \dots \quad (16)$$

$$KT^4_4 = \frac{\partial^2\alpha}{\partial r^2} + \frac{3}{r}\frac{\partial\alpha}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2\alpha}{\partial\theta^2} + \cot\theta\frac{\partial\alpha}{\partial\theta}\right) \quad \ldots \quad (17)$$

$$-KT_2^1 = -r^2KT_1^2 = \tfrac{1}{2}\frac{\partial^2\alpha}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\alpha}{\partial\theta} \quad \ldots \quad (18)$$

$$-KT^1_4 = +KT^4_1 = \tfrac{1}{2}\frac{\partial^2\alpha}{\partial r\partial t} \quad \ldots \quad (19)$$

and
$$-r^2KT_4^2 = KT_4^2 = \frac{\partial^2\alpha}{\partial\theta\partial t} - \tfrac{1}{2}\cot\theta\frac{\partial\alpha}{\partial t} \quad \ldots \quad (20)$$

Only first-order infinitesimals are retained in these expressions as α and its derivatives are of the first order. It is also considered that at the point of interest which is at a large distance from the rotating masses the space-time is symmetrical and conditions of isotropy are also satisfied. We have, therefore, $T_1^1 = T_2^2 = T_3^3$, which give $\frac{\partial\alpha}{\partial r} = 0$. Also, if $T_2^1 = 0 = T_1^2$, $\frac{\partial\alpha}{\partial\theta} = 0$. These latter three relations give $\alpha = 4\chi^2 + \text{constant}$ ... (21)

so that the metric will be given

$$ds^2 = -(1+\chi^2)(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+(1-4\chi^2)dt^2 \quad \ldots \quad (22)$$

As χ^2 is a function of t only, the transformed metric is;

$$ds^2 = -\{1+4[x(t^*)]^2\}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+dt^{*2} \quad \ldots \quad (22')$$

3 ANOTHER FORM OF METRIC

The condition that the pressure should be isotropic gives

$$-KT_\nu^\mu = \begin{vmatrix} \frac{\partial^2\alpha}{\partial t^2} & 0 & 0 & 0 \\ 0 & \frac{\partial^2\alpha}{\partial t^2} & 0 & \frac{\cot^2\theta}{2r^2}\cdot\frac{\partial\alpha}{\partial t} \\ 0 & 0 & \frac{\partial^2\alpha}{\partial t^2} & 0 \\ 0 & -\frac{\cot\theta}{2}\frac{\partial\alpha}{\partial t} & 0, & 0 \end{vmatrix} \quad \ldots \quad (23)$$

where $\alpha = 4\chi^2 + \text{constant}$; $\chi^2 = \frac{1}{1-R_0^2\sin^2\theta_0\Omega^2}$; $\frac{\partial\alpha}{dt} = 8R_0^2\sin^2\theta\chi^4\Omega\dot{\Omega}$; and

$$\frac{\partial^2\alpha}{\partial t^2} = 8\dot{R}_0^2\sin^2\theta\chi^4\{\Omega\ddot{\Omega} - 3\dot{\Omega}^2 + 4\dot{\Omega}^2\chi^2\}$$

The density ρ_{00} turns out to be zero to a first approximation as would naturally be expected. We may however consider that the condition of isotropy is not strictly satisfied and that $T_2^{\,1} = 0 = T_1^{\,2}$, then, a particular so.ution can be found viz.,

$$\alpha = m(1-r^2 \sin^2\theta \,.\, \omega^2) \qquad \ldots \quad (24)$$

where m is an infinitesimal constant and we may take to depend upon time. Obviously ω may be considered as the angular velocity. The energy-momentum tensor can now be written as

$$-KT_{\mu\nu} = \begin{vmatrix} 2m\sin^2\theta\{\omega^2+r^2(\omega\ddot{\omega}+\dot{\omega}^2), & 0 & 0 & 2mr\sin^2\theta\omega\dot{\omega} \\ 0 & -2mr^2\sin^2\theta\{\omega^2-r^2(\omega\ddot{\omega}+\dot{\omega}^2), & 0 & 3mr^2\sin\theta\cos\theta\omega\dot{\omega} \\ 0 & 0, & -2mr^2\sin^4\theta\{\omega^2-r^2(\omega\ddot{\omega}+\dot{\omega}^2). & 0 \\ 2mr\sin^2\theta\omega\dot{\omega}, & 3mr^2\sin\theta\cos\theta\omega\dot{\omega}, & 0, & 2m\{\omega^2+r^2\sin^2\theta(\omega\ddot{\omega}+\dot{\omega}^2)\} \end{vmatrix} \qquad (25)$$

The mechanical relation which must be satisfied by the energy-momentum tensor is given by $\dfrac{\partial T^{\mu\nu}}{\partial x^\nu} = 0$. It will be seen that this gives $\dfrac{\partial T^{33}}{\partial\phi} = 0$, which is satisfied n the present case. The other relations are :

$$\left.\begin{aligned} &\frac{\partial T^{11}}{\partial r} + \frac{\partial T^{14}}{\partial t} = 0 \\ &\frac{\partial T^{22}}{\partial\theta} + \frac{\partial T^{24}}{\partial t} = 0 \\ \text{and}\quad &\frac{\partial T^{44}}{\partial t} + \frac{\partial T^{41}}{\partial r} + \frac{\partial T^{42}}{\partial\theta} = 0 \end{aligned}\right\} \qquad \ldots \quad (26)$$

which gives a differential equation for the determination of ω namely,

$$\omega\ddot{\omega}+a\dot{\omega}^2+b\omega^2 = 0 \qquad \ldots \quad (27)$$

where a and b are constant coefficients.

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